4/25/01

Lecture 40 Fine Structure of H

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \qquad O(\alpha^2 mc^2)$$

$$H' = \left.\begin{array}{l}\text{Relativistic} \\ + \text{ Spin Orbit}\end{array}\right\} \qquad O(\alpha^4 mc^2)$$

$+$ Lamb shift $\qquad O(\alpha^5 mc^2)$

$+$ Hyperfine $\qquad O\left[\alpha^4 \left(\frac{m}{m_p}\right) mc^2\right]$

Relativistic

$$H'_r = \sqrt{p^2c^2 + m^2c^4} - mc^2 - \frac{p^2}{2m} \quad (\text{in } H_0)$$

$$= -\frac{p^4}{8m^3c^2}$$

First order correction to energy

$$E_r^1 = \langle \psi_{nlm} | H_r^1 \psi_{nlm} \rangle$$

$$= -\frac{1}{8m^3c^2} \langle \psi_{nlm} | p^4 \psi_{nlm} \rangle$$

$$= -\frac{1}{8m^3c^2} \langle p^2\psi | p^2\psi \rangle$$

$$\hat{H}_0 \psi = E_n^0 \psi \implies \hat{p}^2\psi = 2m(E_n^0 - \hat{V})\psi$$

$$E_r^1 = -\frac{1}{2mc^2}\left[{E_n^0}^2 - 2E_n^0\langle \hat{V}\rangle + \langle \hat{V}^2\rangle\right]$$

For $\langle V \rangle$ need $\langle \frac{1}{r} \rangle = \frac{1}{n^2 a}$

$$\psi_{n l m} = N_{n l}\, e^{-r/na} \left(\frac{2r}{na}\right)^{l} L^{2l+1}_{n-l-1}\left(\frac{2r}{na}\right) Y^{m}_{l}(\theta, \phi)$$

$$\langle \psi_{n l m} | \frac{1}{r^2} \psi_{n l m} \rangle = \frac{1}{(l+\frac{1}{2}) n^3 a^2}$$ much work

$$E^0_n = -\frac{mc^2}{2} \frac{\alpha^2}{n^2}$$

$$a = 4\pi \epsilon_0 \hbar^2 / m e^2$$

$$\boxed{E^1_r = -\frac{E^{0\,2}_n}{2mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right]}$$

<u>Spin orbit interaction</u>

In electrons rest frame, moving proton creates magnetic field

$$\vec{B} = \frac{1}{4\pi \epsilon_0} \frac{e}{mc^2 r^3} \vec{L}$$

$$H = -\vec{\mu} \cdot \vec{B}$$

$$\mu = -\frac{ge}{2m_e} \vec{S} \qquad g \simeq 2$$

$$H = \left(\frac{e^2}{4\pi\epsilon_0}\right) \frac{1}{m^2 c^2 r^3} \vec{S} \cdot \vec{L}$$

But note election is not an inertial frame. Correction from Thomas precession (rel. not QM) reduces H by $\frac{1}{2}$

$$H'_{so} = \frac{e^2}{8\pi\epsilon_0} \frac{1}{m^2 c^2 r^3} \vec{S} \cdot \vec{L}$$

For $\langle H'_{so} \rangle$ need $\langle \frac{1}{r^3} \rangle$ and $\langle \vec{S} \cdot \vec{L} \rangle$

$$\langle \frac{1}{r^3} \rangle = \frac{1}{l(l+\frac{1}{2})(l+1) n^3 a^3}$$ (prob. 6.30)

$$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{J}^2 = L^2 + S^2 + 2L \cdot S$$

$$\vec{L} \cdot \vec{S} = \frac{1}{2}\left[\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right]$$

$$\langle L \cdot S \rangle = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+\tfrac{1}{2})\right]$$

with $s = \frac{1}{2}$

Example $\ell = 1$ $\quad s = \frac{1}{2}$

$j = \frac{1}{2}, \frac{3}{2}$

$$\langle L \cdot S \rangle_{j=\frac{1}{2}} = \frac{\hbar^2}{2}\left(\frac{3}{4} - 2 - \frac{3}{4}\right) = -\hbar^2$$

$$\langle L \cdot S \rangle_{j=\frac{3}{2}} = \frac{\hbar^2}{2}\left(\frac{3}{2}\,\frac{5}{2} - 2 - \frac{3}{4}\right) = \frac{\hbar^2}{2}$$

$$E^1_{so} = \frac{E_n^2}{mc^2} \frac{n\left(j(j+1) - \ell(\ell+1) - 3/4\right)}{\ell\,(\ell+\frac{1}{2})\,(\ell+1)}$$

$$E^1_{fs} = E^1_r + E^1_{so} = \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right)$$

$$E_{nj} = E^0_n + E^1_{fs} = -\frac{13.6\,eV}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$