4/13/01

Lecture 35 Fermi Gas

Gas of nearly free elections

Model for
- Metalic H
- Valence elections in metal
- White Dwarf star

Fermi gas of neutrons $\Rightarrow$ neutron star

Fermi gas of quarks $\rightarrow$ quark gluon plasma

Conside elections in a 3-Dim Square well

$$V = \begin{cases} 0 & 0<x<l, \quad 0<y<l, \quad 0<z<l \\ \infty & \text{else} \end{cases}$$

$$\psi_{n_x n_y n_z}(x,y,z) = \sqrt{\frac{8}{V}} \sin\left(\frac{n_x \pi x}{l}\right) \sin \frac{n_y \pi y}{l} \sin\left(\frac{n_z \pi z}{l}\right)$$

$V = l^3$ Volume

Normalization $\int_0^l dx \int_0^l dy \int_0^l dz \, |\psi|^2 = 1$

$k^2 = k_x^2 + k_y^2 + k_z^2$ Wave vector

$$k_i = n_i \frac{\pi}{l}$$

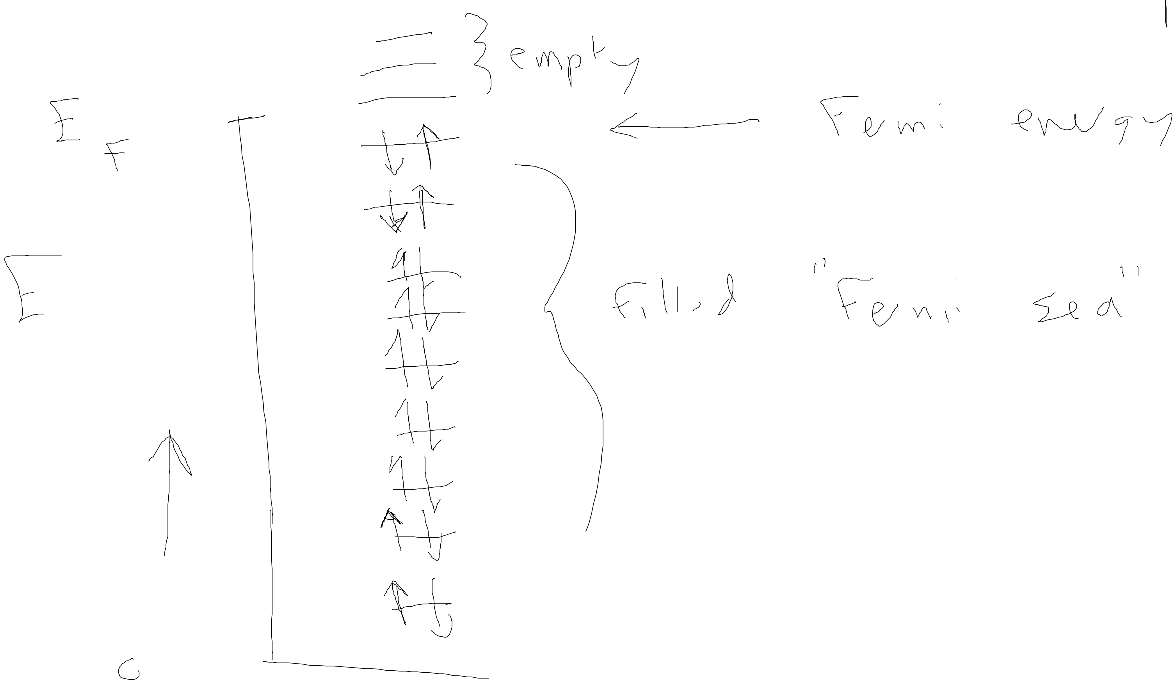
momentum $p = \hbar k$

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2ml^2}(n_x^2 + n_y^2 + n_z^2)$$

$n_x, n_y, n_z = 1, 2, 3, \ldots \infty$

Lowest energy for system fills up all levels below some maximum energy called the Fermi energy.

States above the Fermi level are empty.



Because of Pauli exclusion principle each level can have at most 2 electrons, one spin up, one spin down

Let system have a density of ρ electrons per volume

$$\rho = \frac{N_{tot}}{V}$$

$$N_{tot} = \sum_{n_x n_y n_z = 1}^{E(n_x, n_y, n_z) < E_F} 2$$

$$E(n_x, n_y, n_z) = \frac{\hbar^2 \pi^2}{2 m l^2} (n_x^2 + n_y^2 + n_z^2)$$

Number of terms is very large

$N_{tot} \sim 6 \times 10^{23}$ avagad. #

Replace sum by integral

$$N_{tot} \simeq 2 \int_{n_x, n_y, n_z > 0}^{E(\vec{n}) < E_F} d^3 n$$

$$\int_1^{n_{max}} dn_x \simeq \int_0^{n_{max}} dn_x = \frac{1}{2} \int_{-n_{max}}^{n_{max}} dn_x$$

$$N_{tot} \simeq 2 \left(\frac{1}{2}\right)^3 \int^{E(\vec{n}) < E_F} d^3 n$$

$$E(n) = \frac{\hbar^2 \pi^2}{2 m l^2} n^2$$

let $\vec{k} = \frac{\pi}{l} \vec{n}$

$$\left(\frac{l}{\pi}\right)^3 d^3 k = d^3 n$$

$$N_{tot} = 2 \left[\frac{V}{(2\pi)^3}\right] \int^{E(k) < E_F} d^3 k$$

$$N_{tot} = 2 \frac{V}{(2\pi)^3} 4\pi \int_0^{k_F} k^2 \, dk$$

$$\frac{N_{tot}}{V} = \rho = \frac{2}{(2\pi)^3} \frac{4\pi}{3} k_F^3 = \frac{k_F^3}{3\pi^2}$$

$$E(k) \leq E_F = \frac{\hbar^2 k_F^2}{2m}$$

$k_F = \frac{\sqrt{2 m E_F}}{\hbar}$ Fermi wave vector

$p_F = \hbar k_F = \sqrt{2 m E_F}$ Fermi momentum

Fermi "momentum"

$$\boxed{k_F = \left[3\pi^2 \rho\right]^{1/3}}$$

All states with wave vector below k_F are filled, all states with wave vector above k_F are empty (at zero T)

Fermi Energy

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2)^{2/3} \rho^{2/3}$$

Total Energy of gas

$$E_{tot} = 2 \sum_{n_x n_y n_z = 1}^{E(n) < E_F} \frac{\hbar^2 k^2}{2m}$$

$$= 2 \frac{V}{(2\pi)^3} \int^{k_F} d^3k \, \frac{\hbar^2 k^2}{2m}$$

$$= \frac{V}{\pi^3} \int_0^{k_F} k^2 \, dk \, \frac{\hbar^2 k^2}{2m}$$

$$= \frac{\hbar^2}{10m} \frac{V}{\pi^3} k_F^5$$

$$= \frac{\hbar^2}{10m\pi^2} V (3\pi^2)^{5/3} \frac{N_{tot}^{5/3}}{V^{5/3}}$$

$$\boxed{E_{tot} = \frac{\hbar^2}{10m\pi^2}(3\pi^2)^{5/3} N_{tot}^{5/3} \frac{1}{V^{2/3}}}$$

1^{st} law $dE = dq + dw$

$$dW = -P\,dV \Rightarrow \boxed{P = -\left(\frac{dE}{dV}\right)_n}$$

$$P = \frac{\hbar^2}{10 m \pi^2} (3\pi^2)^{5/3} N_{tot}^{5/3} \frac{2}{3} \frac{1}{V^{5/3}}$$

$$= \frac{\hbar^2}{15 m \pi^2} (3\pi^2)^{5/3} \rho^{5/3}$$

$$\boxed{P = \frac{\hbar^2}{5 m} (3\pi^2)^{2/3} \rho^{5/3}}$$

Fermi pressure or degeneracy pressur
Even at $T=0$ Fermi gas still exerts this pressure because of zero point motion as elections bounce off of walls