4/6/01

Lecture 32 Addition of Ang. Mom. Cont.

Add angular momentum ℓ (orbital) to spin $\frac{1}{2}$ to get total angular momentum J

$$|JM\rangle = \sum_{m_\ell m_s} C^{\ell \frac{1}{2} J}_{m_\ell m_s M} Y_\ell^{m_\ell} \chi_{m_s}$$

If M is the maximum possible value

$$M = \ell + \frac{1}{2}$$

Only one way to make it

$$m_\ell = \ell \qquad m_s = +\frac{1}{2}$$

also add $s = \frac{1}{2}$ to ℓ allowed J values

$$|\ell - \tfrac{1}{2}| \le J \le \ell + \tfrac{1}{2}$$

and $J \ge |M| \implies J = M = \ell + \frac{1}{2}$

$$|\ell+\tfrac{1}{2}, \ell+\tfrac{1}{2}\rangle = Y_\ell^\ell \chi_{+\frac{1}{2}}$$

so

$$C^{\ell \frac{1}{2}, \ell+\frac{1}{2}}_{\ell, \frac{1}{2}, \ell+\frac{1}{2}} = \underline{1}$$

Now apply $J_- = L_- + S_-$

$$J_- |JM\rangle = A^M_J |J, M-1\rangle$$

From problem (4.19)

$$A^M_J = \hbar\sqrt{J(J+1) - M(M-1)}$$

$$J = M = \ell + \tfrac{1}{2}$$

$$A^M_J = \hbar\sqrt{(\ell+\tfrac{1}{2})(\ell+\tfrac{3}{2}) - (\ell+\tfrac{1}{2})(\ell-\tfrac{1}{2})} = \hbar\sqrt{2\ell+1}$$

$$A_J^m \, |J, J-1\rangle = A_l^l \, Y_l^{l-1} \chi_{1/2} + A_s^{1/2} Y_l^l \chi_{-1/2}$$

$$A_l^l = \hbar\sqrt{l(l+1) - l(l-1)} = \hbar\sqrt{2l}$$

$$A_s^{1/2} = \hbar\sqrt{\tfrac{1}{2}(3/2) - \tfrac{1}{2}(-\tfrac{1}{2})} = \hbar$$

$$\hbar\sqrt{2l+1}\, |J, J-1\rangle = \hbar\left(\sqrt{2l}\, Y_l^{l-1} \chi_{1/2} + Y_l^l \chi_{-1/2}\right)$$

so

$$|\underset{l+1/2}{J}, \underbrace{J-1}_{l-1/2}\rangle = \sqrt{\frac{2l}{2l+1}}\, Y_l^{l-1} \chi_{1/2} + \frac{1}{\sqrt{2l+1}} Y_l^l \chi_{-1/2}$$

$$\Rightarrow \quad C^{l,\frac{1}{2},l+\frac{1}{2}}_{l-1,\frac{1}{2},l-\frac{1}{2}} = \sqrt{\frac{2l}{2l+1}}, \quad C^{l,\frac{1}{2},l+\frac{1}{2}}_{l,-\frac{1}{2},l-\frac{1}{2}} = \frac{1}{\sqrt{2l+1}}$$

State $|J, J-1\rangle$, $Y_l^{l-1}\chi_{1/2}$ ... are normalized

$$\Rightarrow \sum_{m_l m_s} \left| C^{l\ \frac{1}{2},\ l+\frac{1}{2}}_{m_l\ m_s\ m_l+m_s} \right|^2 = 1$$

$$= \frac{2l}{2l+1} + \frac{1}{2l+1} = 1 \quad \checkmark$$

Identical Particles

Start reading Chapter 5

Two particle system $\Psi(\vec{r}_1, \vec{r}_2; t)$

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

$$\hat{H} = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2)$$

Normalization

$$P(r_1, r_2) = |\Psi(\vec{r}_1, \vec{r}_2, t)|^2 \; d^3 r_1 \; d^3 r_2$$

is simultaneous prob. to find 1 within $d^3 r_1$ at $\vec{r}_1$ and at same time to find 2 within $d^3 r_2$ of $\vec{r}_2$.

So

$$\int |\Psi(\vec{r}_1, \vec{r}_2, t)|^2 \; d^3 r_1 \; d^3 r_2 = 1$$

Six dimensional integral

Time independant Schrodinger eq. as before

$$\Psi(r_1, r_2, t) = \psi(\vec{r}_1, \vec{r}_2) \, e^{-iEt/\hbar}$$

E = total energy of system

$$\hat{H} \psi(\vec{r}_1, \vec{r}_2) = E \psi(\vec{r}_1, \vec{r}_2)$$

$$\hat{H} = -\frac{\hbar^2}{2m_1} \nabla_1^2 - \frac{\hbar^2}{2m_2} \nabla_2^2 + V(\vec{r}_1, \vec{r}_2)$$

In general expect complex correlations between particles. For a two electron (He) atom ~~because~~ electrons repell each other expect

$$\psi \to 0 \quad \text{as} \quad |\vec{r_1 - r_2}| \to 0$$

As a simple case, consider small interactions between particles.

Example He like Uranium ion.

Uranium has $Z = 92$ (Nucleus has a charge of 92) If one has a Uranium atom that has been ionized 90 times so only two electrons are left

$$V(\vec{r}_1, \vec{r}_2) = -\frac{Ze^2}{4\pi\epsilon_0 r_1} - \frac{Ze^2}{4\pi\epsilon_0 r_2} + \frac{e^2}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_2|}$$

$Z = 92. \Rightarrow$ can neglect e-e repulsion $\frac{e^2}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_2|}$ term compared to the large attraction of each electron to the nucleus.

If $V(\vec{r}_1, \vec{r}_2) \approx V(r_1)\, V(r_2)$ look for a solution

$$\psi(r_1, r_2) \approx \psi_a(\vec{r}_1)\, \psi_b(\vec{r}_2)$$

seperation of variables

This assumes one can tell the two electrons apart.

In classical mech. you can put a red dot on one of the electrons to tell it apart from the other.

In quantum mechanics all electrons are exactly identical. You would have to make the "red dot" out of some other particle but then you would no longer have just an electron.

Quantum mechanics can deal with particles that are <u>indistinguishable</u> even in principle.

Perhaps an additional extension of HUP I can't make a measurement that tells which election is which.

If I don't know that 1 was in state a and 2 state b or the other way around consider combination

$$\psi_{\pm}(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right]$$

Observed property

All integer spin particles, called <u>bosons</u>, are symmetric under interchange $1 \leftrightarrows 2$

All ½ integer spin particles, called <u>Fermions</u>, are antisymmetric under $1 \leftrightarrows 2$

<u>He like Uranium atom</u>

$$H_1 = -\frac{\hbar^2}{2m_1}\nabla_1 - \frac{Ze^2}{4\pi\epsilon_0 r_1}, \qquad H_2 = -\frac{\hbar^2\nabla_2^2}{2m_2} - \frac{Ze^2}{4\pi\epsilon_0 r_2}$$

$$H_1\psi_a(r_1) = E_a\psi(r_1)$$

$$H_2\psi_b(r_2) = E_b\psi(r_2)$$

$$\psi = A\left[\psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1)\right]$$

$$E \cancel{=} E_a + E_b$$

because elections are fermions

<u>Spin statistics theorem</u>

all integer spin particles are bosons (photons $(s=1)$, mesons $(s=0,1)$

all half integer spin particles are Fermions

electrons, quarks, ...

<u>Pauli exclusion principle</u>

Two identical Fermion particles can not be in the same spatial and spin state!

$$A\left[\psi_a(r_1)\,\psi_a(r_2) - \psi_a(r_2)\,\psi_a(r_1)\right] \equiv 0$$