4/4/01

Lecture 31 Addition of Angular Momenta

Consider two spin ½ particles
Each can be spin up or down

Most general wave function is a linear combination of 4 states

$\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, $\downarrow\downarrow$

spin of 1 spin of 2nd particle

What is total spin S^2 and z projection S_z of these states?

$$S_z = \underbrace{S_z^{(1)}}_{\text{Operator for particle 1}} + S_z^{(2)}$$

Consider <u>direct product</u> state

$$|s_1 m_1 s_2 m_2\rangle = |s_1 m_1\rangle |s_2 m_2\rangle$$

with $s_1 = s_2 = \frac{1}{2}$ and $m_1 = \pm\frac{1}{2}$, $m_2 = \pm\frac{1}{2}$

$$S_z |\tfrac{1}{2} m_1\rangle|\tfrac{1}{2} m_2\rangle = (S_z^{(1)} + S_z^{(2)})|\tfrac{1}{2} m_1\rangle|\tfrac{1}{2} m_2\rangle$$

$$= \hbar (m_1 + m_2) |\tfrac{1}{2} m_1\rangle |\tfrac{1}{2} m_2\rangle$$

Total z projection is just sum of individual z projections

$$m_{tot} = m_1 + m_2$$

$$S^{(2)} = (\vec{S}^{(1)} + \vec{S}^{(2)}) \cdot (\vec{S}^{(1)} + \vec{S}^{(2)})$$

$$= S^{(1)2} + S^{(2)2} + 2\, \vec{S}^{(1)} \cdot \vec{S}^{(2)}$$

Note $[S^{(1)}, S^{(2)}] = 0$ operators act on different particles

$$S^{(i)2}\,|\tfrac{1}{2}\,m_s\rangle = \frac{3}{4}\hbar^2\,|\tfrac{1}{2}\,m_s\rangle = \hbar^2\, s(s+1)\,|\tfrac{1}{2}\,m_s\rangle$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_x\uparrow = \frac{\hbar}{2}\downarrow \qquad S_x\downarrow = \frac{\hbar}{2}\uparrow$$

$$S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad S_y\uparrow = \frac{\hbar}{2}\,i\downarrow \qquad S_y\downarrow = -i\frac{\hbar}{2}\uparrow$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad S_z\uparrow = \frac{\hbar}{2}\uparrow \qquad S_z\downarrow = -\frac{\hbar}{2}\downarrow$$

$$S^2 = \frac{3}{2}\hbar^2 + 2\,\vec{S}^{(1)}\cdot\vec{S}^{(2)}$$

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)} = S_x^{(1)}S_x^{(2)} + S_y^{(1)}S_y^{(2)} + S_z^{(1)}S_z^{(2)}$$

Consider state $\uparrow\uparrow$

$$S_x^{(1)}S_x^{(2)}\uparrow\uparrow = \frac{\hbar^2}{4}\downarrow\downarrow$$

$$S_y^{(1)}S_y^{(2)}\uparrow\uparrow = \frac{\hbar^2}{4}\,i^2\downarrow\downarrow = -\frac{\hbar^2}{4}\downarrow\downarrow$$

$$S_z^{(1)}S_z^{(2)}\uparrow\uparrow = \frac{\hbar^2}{4}\uparrow\uparrow$$

so

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}\uparrow\uparrow = \frac{\hbar^2}{4}\uparrow\uparrow$$

$$S^2\uparrow\uparrow = \left(\frac{3}{2}\hbar^2 + 2\,\frac{\hbar^2}{4}\right)\uparrow\uparrow$$

$$= 2\hbar^2\uparrow\uparrow$$

$$= s_{tot}(s_{tot}+1)\,\hbar^2\uparrow\uparrow$$

with $s_{tot} = 1$

$\uparrow\uparrow$ is an eigenstate of S^2 with $s_{tot} = 1$

write

$\uparrow\uparrow = |\tfrac{1}{2} + \tfrac{1}{2}\rangle |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = |1\ 1\rangle$ spin 1 $S_z = 1$

likewise

$\downarrow\downarrow = |\tfrac{1}{2} - \tfrac{1}{2}\rangle |\tfrac{1}{2} - \tfrac{1}{2}\rangle = |+1 -1\rangle$ spin 1 $S_z = -1$

because

$\vec{S}^{(1)} \cdot \vec{S}^{(2)} \downarrow\downarrow = \frac{\hbar^2}{4} \downarrow\downarrow$

so $S^2 \downarrow\downarrow = 2\hbar^2 \downarrow\downarrow$

For a spin 1 state expect 3 states $M = -1, 0, 1$

We have found $M = -1$ and 1 expect also an $M = 0$ state.

Since $M = m_1 + m_2$ clearly only two possibilities

$\uparrow\downarrow$ or $\downarrow\uparrow$

Expect $S = 1$, $M = 0$ to be a linear combination of these two.

$S_x^{(1)} S_x^{(2)} \uparrow\downarrow = \frac{\hbar^2}{4} \downarrow\uparrow$ spins flipped

$S_y^{(1)} S_y^{(2)} \uparrow\downarrow = \frac{\hbar^2}{4} (i)(-i) \downarrow\uparrow = \frac{\hbar^2}{4} \downarrow\uparrow$

$S_z^{(1)} S_z^{(2)} \uparrow\downarrow = -\frac{\hbar^2}{4} \uparrow\downarrow$ not flipped

so $\vec{S}^{(1)} \cdot \vec{S}^{(2)} \uparrow\downarrow = \frac{\hbar^2}{4} (2\downarrow\uparrow - \uparrow\downarrow)$

not an eigenstate.

likewise

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}\downarrow\uparrow = \frac{\hbar^2}{4}\left(2\uparrow\downarrow - \downarrow\uparrow\right)$$

Consider

$$\frac{1}{\sqrt{2}}\left(\uparrow\downarrow + \downarrow\uparrow\right)$$

$$S^{(1)}\cdot S^{(2)}\left(\uparrow\downarrow + \downarrow\uparrow\right) = \frac{\hbar^2}{4}\left(2\downarrow\uparrow - \uparrow\downarrow + 2\uparrow\downarrow - \downarrow\uparrow\right)$$

$$= \frac{\hbar^2}{4}\left(\uparrow\downarrow + \downarrow\uparrow\right) \quad \text{eigenstate}$$

so

$$S^2\left(\uparrow\downarrow + \downarrow\uparrow\right) = \left(\frac{3\hbar^2}{2} + 2\frac{\hbar^2}{4}\right)\left(\uparrow\downarrow + \downarrow\uparrow\right)$$

$$= 2\hbar^2\left(\uparrow\downarrow + \downarrow\uparrow\right)$$

Normalize state

$$\left(\uparrow\downarrow + \downarrow\uparrow\right)^{+}\left(\uparrow\downarrow + \downarrow\uparrow\right)$$

$$= \underbrace{\uparrow^{+}\uparrow\downarrow^{+}\downarrow}_{1} + \underbrace{\downarrow^{+}\uparrow\uparrow^{+}\downarrow}_{0} + \underbrace{\uparrow^{+}\downarrow\downarrow^{+}\uparrow}_{0} + \underbrace{\downarrow^{+}\downarrow\uparrow^{+}\uparrow}_{1}$$

$$= 2$$

Note $\uparrow^{+}\uparrow = 1$, $\uparrow^{+}\downarrow = 0$

so normalized state $\frac{1}{\sqrt{2}}\left(\uparrow\downarrow + \downarrow\uparrow\right) = |1\,0\rangle$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle$$

$$= \sum_{m_1 m_2} C^{s_1 s_2 S}_{m_1 m_2 M}\,|s_1\,m_1\rangle\,|s_2\,m_2\rangle$$

$S=1$, $M=0$
$s_1 = s_2 = \frac{1}{2}$ and $C^{s_1 s_2 S}_{m_1 m_2 M} = 0$ if $m_1 + m_2 \neq M$

$$C^{\frac{1}{2}\,\frac{1}{2}\,1}_{\frac{1}{2}\,-\frac{1}{2}\,0} = \frac{1}{\sqrt{2}}\,, \qquad C^{\frac{1}{2}\,\frac{1}{2}\,1}_{-\frac{1}{2}\,\frac{1}{2}\,0} = \frac{1}{\sqrt{2}}$$

State $|1\,0\rangle$ is a linear supper position of ~~ss~~ direct product states. In general sum over all allowed m_1 and m_2 subject to $m_1 + m_2 = M$

Finally consider

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}(\uparrow\downarrow - \downarrow\uparrow) = \frac{\hbar^2}{4}\Big[2\downarrow\uparrow - \uparrow\downarrow \cancel{+} - 2\uparrow\downarrow + \downarrow\uparrow\Big]$$

$$= -3\frac{\hbar^2}{4}(\uparrow\downarrow - \downarrow\uparrow)$$

So $$S^2\,\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) = \left[\frac{3}{2}\hbar + 2\left(-\frac{3\hbar^2}{4}\right)\right]\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) = 0$$

eigenstate of S^2 with $S=0$

$$|0\,0\rangle = \sum_{m_1, m_2} C^{\frac{1}{2}\,\frac{1}{2}\,0}_{m_1\,m_2\,0}\,|\tfrac{1}{2}\,m_1\rangle\,|\tfrac{1}{2}\,m_2\rangle$$

$S^2 = 0$ $\quad m_s = 0$

$$C^{\frac{1}{2}\,\frac{1}{2}\,0}_{\frac{1}{2}\,-\frac{1}{2}\,0} = \frac{1}{\sqrt{2}} \quad \text{but} \quad C^{\frac{1}{2}\,\frac{1}{2}\,0}_{-\frac{1}{2}\,\frac{1}{2}\,0} = -\frac{1}{\sqrt{2}}$$

$$(\text{spin } \tfrac{1}{2}) \otimes (\text{spin } \tfrac{1}{2}) = \text{spin } 1 \begin{cases} M = 1 \\ \,0 \\ {-1} \end{cases}$$

$$+ \text{ Spin } 0 \quad M = 0$$

Total of four states $= 2 \times 2$.

$|11\rangle = \uparrow\uparrow$
$|1\,{-1}\rangle = \downarrow\downarrow$
$|10\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$

note all three states are symmetric under interchange of (1) and (2)

$$|00\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

This is antisymmetric combination.

Given two angular momenta

$|j_1 m_1\rangle$ and $|j_2 m_2\rangle$

can chose ~~to~~ diagonalize four operators

Direct product choice $j^2_{(1)}, j^{(1)}_z$
$j^2_{(2)}, j^{(2)}_z$

j can be orbital l or spin s

Or chose total angular momenta

J^2 and J_z also $j^2_{(1)}$ $j^2_{(2)}$

$$|JM\rangle = \sum_{m_1 m_2} C^{j_1 j_2 J}_{m_1 m_2 M} |j_1 m_1\rangle |j_2 m_2\rangle$$

eigenstates are linear combinations of direct product states

$C^{j_1 j_2 J}_{m_1 m_2 M}$ = Clebsch Gordon Coef.

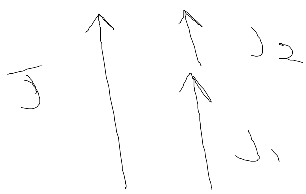
$= \langle j_1 m_1\, j_2 m_2 | JM\rangle$

Allowed values of J range from

$$J = |j_1 - j_2| \text{ to } |j_1 + j_2|$$

in integer steps.

Think of adding two vectors



$J = j_1 + j_2$

$J = j_1 - j_2$

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